OPERATIONS RESEARCH
DOI 10.1287/opre.1090.0714ec

e-companion

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Electronic Companion—"Provably Near-Optimal LP-Based Policies for Revenue Management in Systems with Reusable Resources" by Retsef Levi and Ana Radovanović, *Operations Research*, DOI 10.1287/opre.1090.0714.

Online Appendix

Proof of Lemma 1

Proof: Recall that, in view of discussion above, the probability Q is identical to the blocking probabilities in the corresponding loss network model described above. That is, $Q = \mathbb{P}(\sum_{k=1}^{M'} X_k = C)$.

Next we use the result of Burman et al. (1984), who characterized the stationary distribution for general loss network models (see Theorem 2 in Burman et al. (1984)). In fact, the result of Burman et al. (1984) implies that the stationary distribution of the corresponding loss network model can be expressed through the counterpart system with no capacity constraints. That is, consider an infinite capacity system that faces Poisson streams of requests/customers of class $1, \ldots, M'$, with respective rates $\lambda_1, \ldots, \lambda_{M'}$ and service time distributions (not necessarily exponential) with respective means $\mu_1, \ldots, \mu_{M'}$, and accept *all* the requests/customers. In particular, for each $i = 1, \ldots, M'$, let Y_i be the stationary number of class-i customers being served in the infinite capacity system. Then, for each $n = 0, \ldots, C$, we have $\mathbb{P}\left(\sum_{k=1}^{M'} X_k = n\right) = \frac{\mathbb{P}\left(\sum_{k=1}^{M'} Y_k = n\right)}{\mathbb{P}\left(\sum_{k=1}^{M'} Y_k \leq C\right)}$. Characterizing probability Q using variables $Y_1, \ldots, Y_{M'}$ is very useful since the they are independent of each other, and for each $i = 1, \ldots, M'$, the random variable Y_i follows a Poisson distribution with parameter $\rho_i = \lambda_i \mu_i$. Thus, we get that $P = \frac{\sum_{y \in \mathcal{Y}(C)} \frac{\rho_1^{y_1}}{y_1!} \cdots \frac{\rho_{M'}^{y_{M'}}}{y_{M'}!}}{\sum_{n=0}^{C} \sum_{y \in \mathcal{Y}(n)} \frac{\rho_1^{y_1}}{y_1!} \cdots \frac{\rho_{M'}^{y_{M'}}}{y_{M'}!}}$, where we define $\mathcal{Y}(n) = \{y \in \mathbb{Z}_+^{M'} : \sum_{j=1}^{M'} y_j = n\}$.

Now, by using identity $\sum_{y \in \mathcal{Y}(n)} \frac{\rho_1^{y_1}}{y_1!} \cdots \frac{\rho_{M'}^{y_{M'}}}{y_{M'}!} = \frac{(\rho_1 + \cdots + \rho_{M'})^n}{n!}$, for each $n = 0, \ldots, C$, in conjunction with the expression for the probability Q obtained above, we get $Q = \frac{\frac{(\rho_1 + \cdots + \rho_{M'})^C}{C!}}{\sum_{n=0}^{C} \frac{(\rho_1 + \cdots + \rho_{M'})^n}{n!}}$. Next, consider function $f(z) = \frac{z^C/C!}{\sum_{k=0}^{C} z^k/k!}$. By examining its derivative, it is straightforward to check that $f(z)$ is increasing in z on $(0, C]$. This and Constraint (2) imply that $Q \leq g(C) = \frac{\frac{C^C}{C!}}{\sum_{n=0}^{C} \frac{C^n}{n!}}$. First, observe that $g(1) = 0.5$. To conclude the proof, it is sufficient to show that $g(C)$ is decreasing for all C and that it goes to 0 as C grows to infinity. First, by using a well-known identity (see Section 6.5 of Abramowitz and Stegun (1974)) $\sum_{n=0}^{C} e^{-C} \frac{C^n}{n!} = \frac{\Gamma(C+1, C)}{C!}$ (where $\Gamma(a, x) \triangleq \int_x^{\infty} e^{-t} t^{a-1} dt$ is the incomplete Gamma function), we express

$$g(C+1) = \frac{e^{-C} \frac{C^C}{C!} + [e^{-(C+1)} \frac{(C+1)^{(C+1)}}{(C+1)!} - e^{-C} \frac{C^C}{C!}]}{\sum_{n=0}^{C} e^{-C} \frac{C^n}{n!} + [\sum_{n=0}^{C+1} e^{-(C+1)} \frac{(C+1)^n}{n!} - \sum_{n=0}^{C} e^{-C} \frac{C^n}{n!}]}$$

$$= \frac{e^{-C}\frac{C^C}{C!} + [e^{-(C+1)}\frac{(C+1)^{(C+1)}}{(C+1)!} - e^{-C}\frac{C^C}{C!}]}{\sum_{n=0}^{C} e^{-C}\frac{C^n}{n!} + [\frac{\Gamma(C+2,C+1)}{(C+1)!} - \frac{\Gamma(C+1,C)}{C!}]}. \tag{1}$$

Next, from recurrence $\Gamma(C+2, C+1) = (C+1)\Gamma(C+1, C+1) + (C+1)^{(C+1)}e^{-(C+1)}$ and inequality $\Gamma(C+1, C+1) - \Gamma(C+1, C) \geq -C^C e^{-C}$ (Abramowitz and Stegun (1974)), we derive a lower bound for the second term in the denominator of (4), $\frac{\Gamma(C+2,C+1)}{(C+1)!} - \frac{\Gamma(C+1,C)}{C!}$, by

$$\frac{\Gamma(C+1, C+1) - \Gamma(C+1, C) + (C+1)^C e^{-(C+1)}}{C!} \geq \frac{-C^C e^{-C} + (C+1)^C e^{-(C+1)}}{C!},$$

which implies that $g(C+1) \leq \frac{e^{-C}\frac{C^C}{C!} - [e^{-C}\frac{C^C}{C!} - e^{-(C+1)}\frac{(C+1)^{(C+1)}}{(C+1)!}]}{\sum_{n=0}^{C} e^{-C}\frac{C^n}{n!} - [e^{-C}\frac{C^C}{C!} - e^{-(C+1)}\frac{(C+1)^{(C+1)}}{(C+1)!}]} \leq g(C)$, where the last inequality follows from inequality $\frac{a-x}{b-x} \leq \frac{a}{b}$ that holds for $0 < x < a < b$.

Finally, by utilizing the Stirling approximation, $x! = \sqrt{2\pi}x^{x+1/2}e^{-x+\frac{\theta}{12x}}$, for some $\theta \in (0,1)$ (Abramowitz and Stegun (1974)), we obtain $\frac{\frac{C^C}{C!}}{\sum_{k=0}^{C}\frac{C^k}{k!}} \sim \sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{C}}$. This concludes the proof of the lemma.

Analysis of the price-driven customer arrivals case

For the model with price-driven demand we use the following nonlinear program (NLP1):

$$\max_{\alpha_1,\ldots,\alpha_M,p_1,\ldots,p_M} \sum_{i=1}^{M} p_i \alpha_i \rho_i(p_i) \tag{2}$$

$$\text{s.t.} \sum_{i=1}^{M} \alpha_i \rho_i(p_i) \leq C \tag{3}$$

$$0 \leq \alpha_k \leq 1, \;\; \forall\, 1 \leq i \leq M$$

$$0 \leq p_k, \;\; \forall\, 1 \leq i \leq M.$$

As before, for each $i = 1, \ldots, M$, define $\rho_i(p_i) = \lambda_i(p_i)\mu_i$. In particular, it can be verified that any optimal solution of (NLP1) has only nonnegative prices. Also, observe that for any fixed prices $p_1, \ldots, p_M$, the corresponding solution of $\alpha_1, \ldots, \alpha_M$ has the same knapsack structure defined in Section 2 above. Let $(p^*, \alpha^*) = (p_1^*, \ldots, p_M^*, \alpha_1^*, \ldots, \alpha_M^*)$ be the corresponding optimal solution. Note that if one can solve (NLP1) and obtain the solution (p^*, α^*) then one can construct a similar CSP that will be amenable to the same performance analysis discussed in Section 2.1 above. However, solving (NLP1) directly may be computationally hard. Next, we show that under relatively mild assumptions imposed on the functions $\lambda_1(p_1), \ldots, \lambda_M(p_M)$, one can reduce (NLP1) to an equivalent nonlinear program that is more tractable; we denote it by (NLP2). (By equivalent we mean that they have the same set of optimal solutions.) Consider (NLP2) as follows:

$$\max_{p_1,\ldots,p_M} \sum_{i=1}^{M} p_i \rho_i(p_i) \tag{4}$$

$$\text{s.t.} \sum_{i=1}^{M} \rho_i(p_i) \leq C \tag{5}$$

$$0 \leq p_k, \ \ \forall\, 1 \leq i \leq M.$$

It can be readily verified that as long as $\rho_i(p_i)$ is nonnegative (and decreasing) it is always optimal to have nonnegative prices, so the nonnegativity constraints can be dropped.

LEMMA 1. *The programs (NLP1) and (NLP2) are equivalent.*

Proof: First, we show that for each solution $p = (p_1, \ldots, p_M)$ of (NLP 2), we can construct a solution of (NLP1) with the same objective value. Specifically, consider solution (p', α'), such that $p' = p$ and $\alpha'_i = 1$ if and only if $p_i \rho_i(p_i) > 0$. It can be verified that the resulting solution is feasible for (NLP1) and has the same objective value.

Next, we show how to map optimal solution (p^*, α^*) of (NLP1) to a feasible solution of (NLP2) with the same objective function. For each $i = 1, \ldots, M' - 1$, set $p_i = p_i^*$, and for each $i = M' + 1, \ldots, M$ set $p_i = p_\infty$. It is clear that, for each $i \neq M' - 1$, the resulting contributions to the objective value and Constraint (8) are the same as in (NLP1). Consider now possibly fractional $\alpha_{M'}$. The respective contribution of class M' to the objective value is $\alpha^*_{M'} p^*_{M'} \rho_{M'}(p^*_{M'})$. Similarly, the contribution to Constraint (8) is $\alpha^*_{M'} \rho_{M'}(p^*_{M'})$. Thus, it is sufficient to show that there exists a price $p_{M'}$ such that $p_{M'} \rho_{M'}(p_{M'}) \geq \alpha^*_{M'} p^*_{M'} \rho_{M'}(p^*_{M'})$ and $\rho_{M'}(p_{M'}) \leq \alpha^*_{M'} \rho_{M'}(p^*_{M'})$.

Since $p^*_{M'} \rho_{M'}(p^*_{M'}) \geq \alpha^*_{M'} p^*_{M'} \rho_{M'}(p^*_{M'})$, by the properties of $\lambda_{M'}(p_{M'})$, we know that there exists $\bar{p} \in [p^*_{M'}, P_\infty)$ such that $\bar{p} \rho_{M'}(\bar{p}) = \alpha^*_{M'} p^*_{M'} \rho_{M'}(p^*_{M'})$. Note that $\bar{p} \geq p^*_{M'}$, and, therefore, we obtain $p^*_{M'} \rho_{M'}(\bar{p}) \leq \bar{p} \rho_{M'}(\bar{p}) = \alpha^*_{M'} p^*_{M'} \rho_{M'}(p^*_{M'})$. We conclude that $\rho_{M'}(\bar{p}) \leq \alpha^*_{M'} \rho_{M'}(p^*_{M'})$, which concludes the proof of this lemma.

Lemma 2implies that instead of solving (NLP1) we can, instead, solve (NLP2). However, (NLP2) is computationally more tractable and can be solved relatively easy in many scenarios. Specifically, Lagrangify (dualize) Constraint (8) with some Lagrange multiplier Θ and consider the unconstraint problem $\max_{p_i \in [\Theta, p_\infty)} \sum_{1 \le i \le M} (p_i - \Theta)\rho_i(p_i)$, which is separable in $p_1, \ldots, p_{M'}$. In fact, one aims to find the minimal Θ for which the resulting solution satisfies Constraint (8). This can be done by applying bi-section search on the interval $[0, p_\infty]$. The complexity of this procedure depends on the complexity of maximizing $(p_i - \Theta)\rho_i(p_i)$ for each $1 \le i \le M$. It is not hard to check that there are at least two tractable cases: (i) $\rho_i(p_i)$ is a concave function on $[0, p_\infty)$, for each $1 \le i \le M$; (ii) $\rho_i(p_i)$ is convex, but $p_i\rho_i(p_i)$ is a concave function on $[0, p_\infty)$, for each $1 \le i \le M$.