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Electronic Companion—“A Sampling-Based Approach to Appointment Scheduling” by Mehmet A. Begen, Retsef Levi, and Maurice Queyranne, *Operations Research*, <http://dx.doi.org/10.1287/opre.1120.1053>.

5. Online Appendix - Proofs

Proof. (Polynomial Time Algorithm Theorem 4)

Theorem 7.1 of Begen and Queyranne (2011) implies that $F_{\hat{\mathbf{p}}}(\cdot)$ can be minimized in $O(\sigma(n)EO n^2 \log(\lceil h/2n \rceil))$ where $\sigma(n)$ is the number of function evaluations required to minimize a submodular set function over an n -element ground set and EO is the time needed for an expected cost evaluation. We find the expected cost for $F_{\hat{\mathbf{p}}}(\cdot)$ in $O(nN)$ by computing the total cost for each realization (takes $O(n)$ time) and then taking the average of N total cost realizations, i.e., sample average approximation. Finally, Theorem 4 of Orlin (2007) shows that $\sigma(n) = O(n^5)$. \square

Proof. (Lemma 6)

The proof is by induction on $|\mathcal{F}|$. Let $1 \leq k \leq |\mathcal{F}|$ and

$$Y_k = \{ |Prob_{\mathbf{p}}\{O_k(\mathbf{p})\} - Prob_{\hat{\mathbf{p}}}\{O_k(\mathbf{p})\}| \leq \epsilon' \}.$$

For $\mathcal{F} = 2$ the result holds since

$$\begin{aligned} Prob\{Y_1 \cap Y_2\} &= 1 - Prob\{\overline{Y_1} \cap \overline{Y_2}\} \\ &= 1 - Prob\{\overline{Y_1} \cup \overline{Y_2}\} \\ &\geq 1 - (Prob\{\overline{Y_1}\} + Prob\{\overline{Y_2}\}) \quad (\text{since } Prob\{\overline{Y_1} \cup \overline{Y_2}\} \leq Prob\{\overline{Y_1}\} + Prob\{\overline{Y_2}\}) \\ &\geq 1 - 2\delta' \quad (\text{since } Prob\{\overline{Y_1}\}, Prob\{\overline{Y_2}\} < \delta'). \end{aligned}$$

Suppose the result is true for $|\mathcal{F}| = k$, i.e., $Prob\{\bigcap_{i=1}^k Y_i\} \geq 1 - k\delta'$. Let $Y = \bigcap_{i=1}^k Y_i$. Then,

$$\begin{aligned} Prob\{Y \cap Y_{k+1}\} &= 1 - Prob\{\overline{Y} \cap \overline{Y_{k+1}}\} \\ &= 1 - Prob\{\overline{Y} \cup \overline{Y_{k+1}}\} \\ &\geq 1 - (Prob\{\overline{Y}\} + Prob\{\overline{Y_{k+1}}\}) \quad (\text{as } Prob\{\overline{Y} \cup \overline{Y_{k+1}}\} \leq Prob\{\overline{Y}\} + Prob\{\overline{Y_{k+1}}\}) \\ &\geq 1 - (k+1)\delta' \quad (\text{as } Prob\{\overline{Y}\} < k\delta' \text{ and } Prob\{\overline{Y_{k+1}}\} < \delta'). \end{aligned}$$

Therefore the result is also true for $|\mathcal{F}| = k + 1$, and hence the proof is complete. \square

Proof. (Lemma 7)

Since \widehat{A} is an optimal appointment vector for $F_{\widehat{\mathbf{p}}}$ there exists $\widehat{\mathbf{X}} \in \Theta$ such that $g_k(\widehat{\mathbf{X}}, \widehat{\mathbf{A}})_{\widehat{\mathbf{p}}} = 0$ for all $1 \leq k \leq n + 1$. If $|g_k(\widehat{\mathbf{X}}, \widehat{\mathbf{A}})_{\mathbf{p}} - g_k(\widehat{\mathbf{X}}, \widehat{\mathbf{A}})_{\widehat{\mathbf{p}}}| < \epsilon'K'$ then this implies that $|g_k(\widehat{\mathbf{X}}, \widehat{\mathbf{A}})_{\mathbf{p}} - 0| < \epsilon'K'$, and hence there exists $g \in \partial F_{\mathbf{p}}(\widehat{\mathbf{A}})$ such that $|g_k| < \epsilon'K'$ for all $1 \leq k \leq n + 1$. We now show that $|g_k(\widehat{\mathbf{X}}, \widehat{\mathbf{A}})_{\mathbf{p}} - g_k(\widehat{\mathbf{X}}, \widehat{\mathbf{A}})_{\widehat{\mathbf{p}}}| < \epsilon'K'$. We start by taking the difference $|g_k(\widehat{\mathbf{X}}, \widehat{\mathbf{A}})_{\mathbf{p}} - g_k(\widehat{\mathbf{X}}, \widehat{\mathbf{A}})_{\widehat{\mathbf{p}}}|$ term by term and factoring out $\widehat{\mathbf{X}}$ terms by using equation (2).

$$\begin{aligned}
 |g_k(\widehat{\mathbf{X}}, \widehat{\mathbf{A}})_{\mathbf{p}} - g_k(\widehat{\mathbf{X}}, \widehat{\mathbf{A}})_{\widehat{\mathbf{p}}}| &= \left| \sum_{j=k}^n \alpha_j \sum_{S \in \mathcal{P}^*([j])} \widehat{X}_{kj}^L(S) \left(Prob_{\mathbf{p}}\{I_j = S\} - Prob_{\widehat{\mathbf{p}}}\{I_j = S\} \right) \right. \\
 &\quad - \alpha_{k-1} \sum_{S \in \mathcal{P}^*([k-1])} \left(Prob_{\mathbf{p}}\{I_{k-1} = S\} - Prob_{\widehat{\mathbf{p}}}\{I_{k-1} = S\} \right) \\
 &\quad + \sum_{j=k}^n \beta_j \sum_{S \in \mathcal{P}^*([j])} \widehat{X}_{kj}^{T>}(S) \left(Prob_{\mathbf{p}}\{I_j^> = S\} - Prob_{\widehat{\mathbf{p}}}\{I_j^> = S\} \right) \\
 &\quad - \beta_{k-1} \sum_{S \in \mathcal{P}^*([k-1])} \left(Prob_{\mathbf{p}}\{I_{k-1}^> = S\} - Prob_{\widehat{\mathbf{p}}}\{I_{k-1}^> = S\} \right) \\
 &\quad + \sum_{j=k}^n \beta_j \sum_{S \in \mathcal{P}^*([j])} \widehat{X}_{kj}^{T=}(S) \left(Prob_{\mathbf{p}}\{I_j^= = S\} - Prob_{\widehat{\mathbf{p}}}\{I_j^= = S\} \right) \\
 &\quad - \beta_{k-1} \sum_{S \in \mathcal{P}^*([k-1])} \sum_{i \in S} \widehat{X}_{ik-1}^{T=}(S) \left(Prob_{\mathbf{p}}\{I_{k-1}^= = S\} - Prob_{\widehat{\mathbf{p}}}\{I_{k-1}^= = S\} \right) \\
 &\quad + \sum_{j=k}^n \gamma_j \sum_{S \in \mathcal{P}^*([j])} \widehat{X}_{kj}^{T>}(S) \left(Prob_{\mathbf{p}}\{I_j^> = S\} - Prob_{\widehat{\mathbf{p}}}\{I_j^> = S\} \right) \\
 &\quad - \gamma_{k-1} \sum_{S \in \mathcal{P}^*([k-1])} \left(Prob_{\mathbf{p}}\{I_{k-1}^> = S\} - Prob_{\widehat{\mathbf{p}}}\{I_{k-1}^> = S\} \right) \\
 &\quad + \sum_{j=k}^n \gamma_j \sum_{S \in \mathcal{P}^*([j])} \widehat{X}_{kj}^{M=}(S \cup \{j+1\}) \left(Prob_{\mathbf{p}}\{I_j^= = S\} - Prob_{\widehat{\mathbf{p}}}\{I_j^= = S\} \right) \\
 &\quad \left. - \gamma_{k-1} \sum_{S \in \mathcal{P}^*([k-1])} \left(Prob_{\mathbf{p}}\{I_{k-1}^= = S\} - Prob_{\widehat{\mathbf{p}}}\{I_{k-1}^= = S\} \right) \right|.
 \end{aligned}$$

The term $-\alpha_{k-1} \sum_{S \in \mathcal{P}^*([k-1])} \left(Prob_{\mathbf{p}}\{I_{k-1} = S\} - Prob_{\widehat{\mathbf{p}}}\{I_{k-1} = S\} \right)$ disappears since $\sum_{S \in \mathcal{P}^*([k-1])} Prob_{\mathbf{p}}\{I_{k-1} = S\} = 1 = \sum_{S \in \mathcal{P}^*([k-1])} Prob_{\widehat{\mathbf{p}}}\{I_{k-1} = S\}$. By using the triangular inequality we obtain

$$\begin{aligned}
|g_k(\widehat{\mathbf{X}}, \widehat{\mathbf{A}})_{\mathbf{p}} - g_k(\widehat{\mathbf{X}}, \widehat{\mathbf{A}})_{\widehat{\mathbf{p}}}| &\leq \left| \sum_{j=k}^n \alpha_j \sum_{S \in \mathcal{P}^*([j])} \widehat{X}_{kj}^L(S) \left(\text{Prob}_{\mathbf{p}}\{I_j = S\} - \text{Prob}_{\widehat{\mathbf{p}}}\{I_j = S\} \right) \right| \\
&+ \left| \sum_{j=k}^n \beta_j \sum_{S \in \mathcal{P}^*([j])} \widehat{X}_{kj}^{T>}(S) \left(\text{Prob}_{\mathbf{p}}\{I_j^> = S\} - \text{Prob}_{\widehat{\mathbf{p}}}\{I_j^> = S\} \right) \right| \\
&+ \left| \beta_{k-1} \sum_{S \in \mathcal{P}^*([k-1])} \left(\text{Prob}_{\mathbf{p}}\{I_{k-1}^> = S\} - \text{Prob}_{\widehat{\mathbf{p}}}\{I_{k-1}^> = S\} \right) \right| \\
&+ \left| \sum_{j=k}^n \beta_j \sum_{S \in \mathcal{P}^*([j])} \widehat{X}_{kj}^{T=}(S) \left(\text{Prob}_{\mathbf{p}}\{I_j^= = S\} - \text{Prob}_{\widehat{\mathbf{p}}}\{I_j^= = S\} \right) \right| \\
&+ \left| \beta_{k-1} \sum_{S \in \mathcal{P}^*([k-1])} \sum_{i \in S} \widehat{X}_{ik-1}^{T=}(S) \left(\text{Prob}_{\mathbf{p}}\{I_{k-1}^= = S\} - \text{Prob}_{\widehat{\mathbf{p}}}\{I_{k-1}^= = S\} \right) \right| \\
&+ \left| \sum_{j=k}^n \gamma_j \sum_{S \in \mathcal{P}^*([j])} \widehat{X}_{kj}^{T>}(S) \left(\text{Prob}_{\mathbf{p}}\{I_j^> = S\} - \text{Prob}_{\widehat{\mathbf{p}}}\{I_j^> = S\} \right) \right| \\
&+ \left| \gamma_{k-1} \sum_{S \in \mathcal{P}^*([k-1])} \left(\text{Prob}_{\mathbf{p}}\{I_{k-1}^> = S\} - \text{Prob}_{\widehat{\mathbf{p}}}\{I_{k-1}^> = S\} \right) \right| \\
&+ \left| \sum_{j=k}^n \gamma_j \sum_{S \in \mathcal{P}^*([j])} \widehat{X}_{kj}^{M=}(S \cup \{j+1\}) \left(\text{Prob}_{\mathbf{p}}\{I_j^= = S\} - \text{Prob}_{\widehat{\mathbf{p}}}\{I_j^= = S\} \right) \right| \\
&+ \left| \gamma_{k-1} \sum_{S \in \mathcal{P}^*([k-1])} \left(\text{Prob}_{\mathbf{p}}\{I_{k-1}^= = S\} - \text{Prob}_{\widehat{\mathbf{p}}}\{I_{k-1}^= = S\} \right) \right|. \tag{3}
\end{aligned}$$

We now find an upper bound for $|g_k(\widehat{\mathbf{X}}, \widehat{\mathbf{A}})_{\mathbf{p}} - g_k(\widehat{\mathbf{X}}, \widehat{\mathbf{A}})_{\widehat{\mathbf{p}}}|$ by obtaining an upper bound for each $|\cdot|$ term in equation (3). We do so by using the fact that $\widehat{\mathbf{X}} \in \Theta$ and rewriting some of the probability terms. Note that we will show this for the first and the third terms as the remaining bounds are obtained in a similar manner to either of the first or the third. We start with the first term in equation (3). Let $\mathcal{P}^{\geq}([j]) = \{S \in \mathcal{P}^*([j]) \mid \text{Prob}_{\mathbf{p}}\{I_j = S\} - \text{Prob}_{\widehat{\mathbf{p}}}\{I_j = S\} \geq 0\}$ for $j = 1, \dots, n$ and $\mathcal{P}^{<}([j]) = \{S \in \mathcal{P}^*([j]) \mid \text{Prob}_{\mathbf{p}}\{I_j = S\} - \text{Prob}_{\widehat{\mathbf{p}}}\{I_j = S\} < 0\}$ for $j = 1, \dots, n$. Then by the definition of $\mathcal{P}^{\geq}([\cdot])$ and $\mathcal{P}^{<}([\cdot])$ (equation (4)), the triangular inequality (equation (5) and equation (8)), and the fact that $\widehat{X}_{kj}^L(S) = 0$ if $k \notin S$ (equation (6)) and $0 \leq \widehat{X}_{kj}^L(S) \leq 1$ (equation (7)) we obtain

$$\begin{aligned}
&\left| \sum_{j=k}^n \alpha_j \sum_{S \in \mathcal{P}^*([j])} \widehat{X}_{kj}^L(S) \left(\text{Prob}_{\mathbf{p}}\{I_j = S\} - \text{Prob}_{\widehat{\mathbf{p}}}\{I_j = S\} \right) \right| \\
&= \left| \sum_{j=k}^n \alpha_j \sum_{S \in \mathcal{P}^{\geq}([j])} \widehat{X}_{kj}^L(S) \left(\text{Prob}_{\mathbf{p}}\{I_j = S\} - \text{Prob}_{\widehat{\mathbf{p}}}\{I_j = S\} \right) \right. \\
&\quad \left. + \sum_{j=k}^n \alpha_j \sum_{S \in \mathcal{P}^{<}([j])} \widehat{X}_{kj}^L(S) \left(\text{Prob}_{\mathbf{p}}\{I_j = S\} - \text{Prob}_{\widehat{\mathbf{p}}}\{I_j = S\} \right) \right| \tag{4}
\end{aligned}$$

$$\begin{aligned}
 &\leq \left| \sum_{j=k}^n \alpha_j \sum_{S \in \mathcal{P}^{\geq}([j])} \widehat{X}_{kj}^L(S) \left(\text{Prob}_{\mathbf{p}}\{I_j = S\} - \text{Prob}_{\widehat{\mathbf{p}}}\{I_j = S\} \right) \right| \\
 &\quad + \left| \sum_{j=k}^n \alpha_j \sum_{S \in \mathcal{P}^{<}([j])} \widehat{X}_{kj}^L(S) \left(\text{Prob}_{\mathbf{p}}\{I_j = S\} - \text{Prob}_{\widehat{\mathbf{p}}}\{I_j = S\} \right) \right| \tag{5} \\
 &= \left| \sum_{j=k}^n \alpha_j \sum_{S \in \mathcal{P}^{\geq}([j])} \widehat{X}_{kj}^L(S) \left(\mathbb{1}\{k \in S\} \text{Prob}_{\mathbf{p}}\{I_j = S\} - \mathbb{1}\{k \in S\} \text{Prob}_{\widehat{\mathbf{p}}}\{I_j = S\} \right) \right| \\
 &\quad + \left| \sum_{j=k}^n \alpha_j \sum_{S \in \mathcal{P}^{<}([j])} \widehat{X}_{kj}^L(S) \left(\mathbb{1}\{k \in S\} \text{Prob}_{\mathbf{p}}\{I_j = S\} - \mathbb{1}\{k \in S\} \text{Prob}_{\widehat{\mathbf{p}}}\{I_j = S\} \right) \right| \tag{6} \\
 &\leq \left| \sum_{j=k}^n \alpha_j \sum_{S \in \mathcal{P}^{\geq}([j])} \left(\mathbb{1}\{k \in S\} \text{Prob}_{\mathbf{p}}\{I_j = S\} - \mathbb{1}\{k \in S\} \text{Prob}_{\widehat{\mathbf{p}}}\{I_j = S\} \right) \right| \\
 &\quad + \left| \sum_{j=k}^n \alpha_j \sum_{S \in \mathcal{P}^{<}([j])} \left(\mathbb{1}\{k \in S\} \text{Prob}_{\mathbf{p}}\{I_j = S\} - \mathbb{1}\{k \in S\} \text{Prob}_{\widehat{\mathbf{p}}}\{I_j = S\} \right) \right| \tag{7} \\
 &= \left| \sum_{j=k}^n \alpha_j \left(\text{Prob}_{\mathbf{p}}\{k \in I_j, I_j \in \mathcal{P}^{\geq}([j])\} - \text{Prob}_{\widehat{\mathbf{p}}}\{k \in I_j, I_j \in \mathcal{P}^{\geq}([j])\} \right) \right| \\
 &\quad + \left| \sum_{j=k}^n \alpha_j \left(\text{Prob}_{\mathbf{p}}\{k \in I_j, I_j \in \mathcal{P}^{<}([j])\} - \text{Prob}_{\widehat{\mathbf{p}}}\{k \in I_j, I_j \in \mathcal{P}^{<}([j])\} \right) \right| \\
 &\leq \sum_{j=k}^n \alpha_j \left| \left(\text{Prob}_{\mathbf{p}}\{k \in I_j, I_j \in \mathcal{P}^{\geq}([j])\} - \text{Prob}_{\widehat{\mathbf{p}}}\{k \in I_j, I_j \in \mathcal{P}^{\geq}([j])\} \right) \right| \\
 &\quad + \sum_{j=k}^n \alpha_j \left| \left(\text{Prob}_{\mathbf{p}}\{k \in I_j, I_j \in \mathcal{P}^{<}([j])\} - \text{Prob}_{\widehat{\mathbf{p}}}\{k \in I_j, I_j \in \mathcal{P}^{<}([j])\} \right) \right| \tag{8} \\
 &\leq \sum_{j=k}^n \alpha_j \epsilon' + \sum_{j=k}^n \alpha_j \epsilon' \leq 2\epsilon' \alpha_{\max} n.
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 &\left| \sum_{j=k}^n \beta_j \sum_{S \in \mathcal{P}^*([j])} \widehat{X}_{kj}^{T>}(S) \left(\text{Prob}_{\mathbf{p}}\{I_j^> = S\} - \text{Prob}_{\widehat{\mathbf{p}}}\{I_j^> = S\} \right) \right| \leq 2\epsilon' \beta_{\max} n, \\
 &\left| \sum_{j=k}^n \beta_j \sum_{S \in \mathcal{P}^*([j])} \widehat{X}_{kj}^{T=}(S) \left(\text{Prob}_{\mathbf{p}}\{I_j^= = S\} - \text{Prob}_{\widehat{\mathbf{p}}}\{I_j^= = S\} \right) \right| \leq 2\epsilon' \beta_{\max} n, \\
 &\left| \sum_{j=k}^n \gamma_j \sum_{S \in \mathcal{P}^*([j])} \widehat{X}_{kj}^{T>}(S) \left(\text{Prob}_{\mathbf{p}}\{I_j^> = S\} - \text{Prob}_{\widehat{\mathbf{p}}}\{I_j^> = S\} \right) \right| \leq 2\epsilon' \gamma_{\max} n, \\
 &\left| \sum_{j=k}^n \gamma_j \sum_{S \in \mathcal{P}^*([j])} \widehat{X}_{kj}^{M=}(S \cup \{j+1\}) \left(\text{Prob}_{\mathbf{p}}\{I_j^= = S\} - \text{Prob}_{\widehat{\mathbf{p}}}\{I_j^= = S\} \right) \right| \leq 2\epsilon' \gamma_{\max} n.
 \end{aligned}$$

We now find an upper bound for the third term in equation (3) by using the definition of $I_{k-1}^>$.

$$\left| \beta_{k-1} \sum_{S \in \mathcal{P}^*([k-1])} \left(\text{Prob}_{\mathbf{p}}\{I_{k-1}^> = S\} - \text{Prob}_{\widehat{\mathbf{p}}}\{I_{k-1}^> = S\} \right) \right|$$

$$\begin{aligned}
&= \left| \beta_{k-1} \sum_{S \in \mathcal{P}^*([k-1])} \left(\text{Prob}_{\mathbf{p}}\{I_{k-1} = S \text{ and } P_{i,k-1} > A_k - A_i : i \in S\} \right. \right. \\
&\quad \left. \left. - \text{Prob}_{\hat{\mathbf{p}}}\{I_{k-1} = S \text{ and } P_{i,k-1} > A_k - A_i : i \in S\} \right) \right| \\
&= \left| \beta_{k-1} \sum_{S \in \mathcal{P}^*([k-1])} \sum_{i=1}^{k-1} \left(\text{Prob}_{\mathbf{p}}\{I_{k-1} = S \text{ and } P_{i,k-1} > A_k - A_i\} \mathbb{1}\{i \in S\} \right. \right. \\
&\quad \left. \left. - \text{Prob}_{\hat{\mathbf{p}}}\{I_{k-1} = S \text{ and } P_{i,k-1} > A_k - A_i\} \mathbb{1}\{i \in S\} \right) \right| \\
&= \left| \beta_{k-1} \sum_{i=1}^{k-1} \sum_{S \in \mathcal{P}^*([k-1])} \left(\text{Prob}_{\mathbf{p}}\{I_{k-1} = S \text{ and } P_{i,k-1} > A_k - A_i\} \mathbb{1}\{i \in S\} \right. \right. \\
&\quad \left. \left. - \text{Prob}_{\hat{\mathbf{p}}}\{I_{k-1} = S \text{ and } P_{i,k-1} > A_k - A_i\} \mathbb{1}\{i \in S\} \right) \right| \\
&= \left| \beta_{k-1} \sum_{i=1}^{k-1} \left(\text{Prob}_{\mathbf{p}}\{i \in I_{k-1} \text{ and } P_{i,k-1} > A_k - A_i\} - \text{Prob}_{\hat{\mathbf{p}}}\{i \in I_{k-1} \text{ and } P_{i,k-1} > A_k - A_i\} \right) \right| \\
&= \left| \beta_{k-1} \sum_{i=1}^{k-1} \left(\text{Prob}_{\mathbf{p}}\{i \in I_{k-1}^>\} - \text{Prob}_{\hat{\mathbf{p}}}\{i \in I_{k-1}^>\} \right) \right| \\
&\leq \beta_{k-1} \sum_{i=1}^{k-1} \left| \text{Prob}_{\mathbf{p}}\{i \in I_{k-1}^>\} - \text{Prob}_{\hat{\mathbf{p}}}\{i \in I_{k-1}^>\} \right| \\
&\leq \beta_{k-1} \sum_{i=1}^{k-1} \epsilon' \leq \epsilon' \beta_{\max} n.
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
&\left| \beta_{k-1} \sum_{S \in \mathcal{P}^*([k-1])} \sum_{i \in S} \hat{X}_{i,k-1}^{T=}(S) \left(\text{Prob}_{\mathbf{p}}\{I_{k-1}^- = S\} - \text{Prob}_{\hat{\mathbf{p}}}\{I_{k-1}^- = S\} \right) \right| \leq \epsilon' \beta_{\max} n, \\
&\left| \gamma_{k-1} \sum_{S \in \mathcal{P}^*([k-1])} \left(\text{Prob}_{\mathbf{p}}\{I_{k-1}^> = S\} - \text{Prob}_{\hat{\mathbf{p}}}\{I_{k-1}^> = S\} \right) \right| \leq \epsilon' \gamma_{\max} n, \\
&\left| \gamma_{k-1} \sum_{S \in \mathcal{P}^*([k-1])} \left(\text{Prob}_{\mathbf{p}}\{I_{k-1}^- = S\} - \text{Prob}_{\hat{\mathbf{p}}}\{I_{k-1}^- = S\} \right) \right| \leq \epsilon' \gamma_{\max} n.
\end{aligned}$$

Therefore we can bound $|g_k(\hat{\mathbf{X}}, \hat{\mathbf{A}})_{\mathbf{p}} - g_k(\hat{\mathbf{X}}, \hat{\mathbf{A}})_{\hat{\mathbf{p}}}|$ from above:

$$|g_k(\hat{\mathbf{X}}, \hat{\mathbf{A}})_{\mathbf{p}} - g_k(\hat{\mathbf{X}}, \hat{\mathbf{A}})_{\hat{\mathbf{p}}}| \leq \epsilon' n (2\alpha_{\max} + 6\beta_{\max} + 6\gamma_{\max}) \quad (1 \leq k \leq n+1).$$

Since the cost coefficients (\mathbf{u}, \mathbf{o}) are assumed to be α -monotone we have $0 \leq \alpha_i \leq o_{\max}$, $\beta_i \leq o_{\max}$ and $\gamma_i \leq u_{\max} + o_{\max}$. Therefore $2\alpha_{\max} + 6\beta_{\max} + 6\gamma_{\max} \leq (14o_{\max} + 6u_{\max})$ so we can take $K' = n(14o_{\max} + 6u_{\max})$. We also determine $|\mathcal{F}|$, the maximum number of events we need to compute

$|g_k(\widehat{\mathbf{X}}, \widehat{\mathbf{A}})_{\mathbf{p}} - g_k(\widehat{\mathbf{X}}, \widehat{\mathbf{A}})_{\widehat{\mathbf{p}}}|$ for all k . For each k , we have $(5(n-k+1) + 4(k-1))$ therefore at most $5n$ events. Since $k \leq n+1$ we have $|\mathcal{F}| = (n+1)(5n) = 5n^2 + 5$. This completes the proof. \square

Proof. (Lemma 9)

Fix \mathbf{A} . Let $h(\mathbf{p}) = F(\mathbf{A}|\mathbf{p}) = \sum_{i=1}^n [o_i(C_i - A_{i+1})^+ + u_i(A_{i+1} - C_i)^+]$. We claim that h is convex.

Recall that by Lemma 3.3.1 of Begen (2010), we can rewrite $F(\mathbf{A}|\mathbf{p})$ and hence $h(\mathbf{p})$ as

$$h(\mathbf{p}) = F(\mathbf{A}|\mathbf{p}) = \sum_{i=1}^n \left[\alpha_i(C_i - A_{i+1}) + \beta_i(C_i - A_{i+1})^+ + \gamma_i(\max\{C_i, A_{i+1}\} - \sum_{k=1}^i p_k) \right]$$

for any $\alpha_i \in \mathbb{R}$ ($1 \leq i \leq n$) where $\beta_i = (o_i - \alpha_i)$ and $\gamma_i = [(u_i + \alpha_i) - (u_{i+1} + \alpha_{i+1})]$. Recall that $C_i = \max_{k \leq i} \{A_k + \sum_{t=k}^i p_t\}$ (by the Critical Path Lemma 4.1 of Begen and Queyranne (2011)) so C_i is convex in \mathbf{p} . By α -monotonicity $\alpha_i, \beta_i \geq 0$. Hence the terms $\alpha_i(C_i - A_{i+1})$ and $\beta_i(C_i - A_{i+1})^+$ are convex in \mathbf{p} . Furthermore, the term $\gamma_i(\max\{C_i, A_{i+1}\} - \sum_{k=1}^i p_k)$ is convex in \mathbf{p} . Therefore $h(\mathbf{p})$ is convex.

Recall that $\nu = \min\{u_1, u_2, \dots, u_n, o_1, o_2, \dots, o_n\}$ and \widetilde{C}_i 's are the completion times, but they are deterministic since we are using expected values, \widetilde{p}_i 's, for the processing times. We next show that $F_{\mathbf{p}}(\mathbf{A}) \geq \widetilde{f}(\mathbf{A})$ by applying Jensen's inequality to $h(\mathbf{p})$ and applying Lemma 3.3.1 of Begen (2010) to $F(\mathbf{A}|\widetilde{\mathbf{p}})$:

$$\begin{aligned} F_{\mathbf{p}}(\mathbf{A}) &= E_{\mathbf{p}}[h(\mathbf{p})] \\ &\geq F(\mathbf{A}|\mathbf{E}\mathbf{p}) = F(\mathbf{A}|\widetilde{\mathbf{p}}) = \sum_{i=1}^n \left[\alpha_i(\widetilde{C}_i - A_{i+1}) + \beta_i(\widetilde{C}_i - A_{i+1})^+ + \gamma_i(\max\{\widetilde{C}_i, A_{i+1}\} - \sum_{k=1}^i \widetilde{p}_k) \right] \\ &= \sum_{i=1}^n [o_i(\widetilde{C}_i - A_{i+1})^+ + u_i(A_{i+1} - \widetilde{C}_i)^+] \\ &\geq \sum_{i=1}^n \nu[(\widetilde{C}_i - A_{i+1})^+ + (A_{i+1} - \widetilde{C}_i)^+] = \widetilde{f}(\mathbf{A}). \end{aligned}$$

Next we obtain $\widetilde{\mathbf{A}} \in \arg \min_{\mathbf{A}} \widetilde{f}(\mathbf{A})$. Note that $\widetilde{f}(\mathbf{A}) \geq 0$ for all \mathbf{A} . Set $\widetilde{A}_1 = 0, \dots, \widetilde{A}_{i+1} = \sum_{j=1}^i \widetilde{p}_j, \dots, \widetilde{A}_{n+1} = \sum_{j=1}^n \widetilde{p}_j$, i.e., $\widetilde{A}_{i+1} - \widetilde{A}_i = \widetilde{p}_i$ and $\widetilde{A}_{i+1} = \sum_{k=1}^i \widetilde{p}_k$ for $1 \leq i \leq n$. Then $\widetilde{A}_{i+1} = \widetilde{C}_i$ for all $i = 1, \dots, n$ and $\widetilde{f}(\widetilde{\mathbf{A}}) = 0$. Therefore $\widetilde{\mathbf{A}} = (0, \widetilde{p}_1, \dots, \sum_{j=1}^n \widetilde{p}_j)$ is indeed optimal for \widetilde{f} .

We next show $F_{\mathbf{p}}(\mathbf{A}) \geq \frac{\nu}{n} \|\mathbf{A} - \tilde{\mathbf{A}}\|_1$ by showing $\tilde{f}(\mathbf{A}) \geq \frac{\nu}{n} \|\mathbf{A} - \tilde{\mathbf{A}}\|_1$. Note that $\sum_{i=1}^n [(\tilde{C}_i - A_{i+1})^+ + (A_{i+1} - \tilde{C}_i)^+] = \sum_{i=1}^n |(\tilde{C}_i - A_{i+1})|$, and the result would follow if we show $\sum_{i=1}^j |(\tilde{C}_i - A_{i+1})| \geq |A_{j+1} - \tilde{A}_{j+1}| = |A_{j+1} - \sum_{t=1}^j \tilde{p}_t|$ for all $j = 1, 2, \dots, n$. We now show $\sum_{i=1}^j |(\tilde{C}_i - A_{i+1})| \geq |A_{j+1} - \sum_{t=1}^j \tilde{p}_t|$ for all $j = 1, 2, \dots, n$. We distinguish two cases. First, suppose that $A_{j+1} \leq \sum_{t=1}^j \tilde{p}_t$. Since $\sum_{t=1}^j \tilde{p}_t \leq \tilde{C}_j$, we have $A_{j+1} \leq \tilde{C}_j$. Therefore

$$\begin{aligned} \sum_{i=1}^j |(\tilde{C}_i - A_{i+1})| &\geq |(\tilde{C}_j - A_{j+1})| \\ &\geq \left| \sum_{t=1}^j \tilde{p}_t - A_{j+1} \right| = |\tilde{A}_{j+1} - A_{j+1}| \end{aligned}$$

The second case is where $A_{j+1} > \sum_{t=1}^j \tilde{p}_t$. Then

$$\begin{aligned} \sum_{i=1}^j |(\tilde{C}_i - A_{i+1})| &\geq \sum_{i=1}^j (A_{i+1} - \tilde{C}_i)^+ \\ &= \max(\tilde{C}_j, A_{j+1}) - \sum_{t=1}^j \tilde{p}_t \\ &\geq A_{j+1} - \sum_{t=1}^j \tilde{p}_t = |A_{j+1} - \tilde{A}_{j+1}| \end{aligned}$$

where the first equality follows from a property shown in the proof of Lemma 3.3.1 of Begen (2010).

Hence we obtain

$$\sum_{i=1}^j |(\tilde{C}_i - A_{i+1})| \geq |A_{j+1} - \tilde{A}_{j+1}| \text{ for all } 1 \leq j \leq n.$$

Therefore for every $j = 1, \dots, n$

$$\sum_{i=1}^n |(\tilde{C}_i - A_{i+1})| \geq \sum_{i=1}^j |(\tilde{C}_i - A_{i+1})| \geq |A_{j+1} - \tilde{A}_{j+1}|$$

and hence

$$\begin{aligned} n\tilde{f}(\mathbf{A}) &= n\nu \left(\sum_{i=1}^n [(\tilde{C}_i - A_{i+1})^+ + (A_{i+1} - \tilde{C}_i)^+] \right) \\ &= n\nu \left(\sum_{i=1}^n |(\tilde{C}_i - A_{i+1})| \right) \\ &\geq \nu \|\mathbf{A} - \tilde{\mathbf{A}}\|_1 \end{aligned}$$

as desired. Therefore $F_{\mathbf{p}}(\mathbf{A}) \geq \frac{\nu}{n} \|\mathbf{A} - \tilde{\mathbf{A}}\|_1$. This completes the proof. \square

Proof. **(Lemma 12)**

Let $L = f(y^*)/\lambda$. Consider the norm l_1 ball $B = B(\tilde{y}, L)$, then $y^* \in B(\tilde{y}, L) = \{y : \lambda\|y^* - \tilde{y}\| \leq f(y^*)\}$.

The subgradient inequality at \hat{y} combined with Cauchy-Schwartz inequality yields

$f(\hat{y}) - f(y^*) \leq \alpha\|\hat{y} - y^*\|_1$ (since Cauchy-Schwartz inequality also holds for l_1 norm). We also have

$$\begin{aligned} \|\hat{y} - y^*\|_1 &\leq \|\hat{y} - \tilde{y}\|_1 + \|\tilde{y} - y^*\|_1 \\ &\leq f(\hat{y})/\lambda + L \\ &= f(\hat{y})/\lambda + f(y^*)/\lambda. \end{aligned}$$

So we obtain $f(\hat{y}) - f(y^*) \leq \alpha(f(\hat{y})/\lambda + f(y^*)/\lambda)$ and hence $f(\hat{y}) \leq f(y^*)(\lambda + \alpha)/(\lambda - \alpha)$. If we choose $\alpha \leq \lambda\epsilon/3$ the result follows. \square

Proof. **(Lemma 13)**

If $|g_k| < \epsilon\nu/(3(n+1)n)$ for all $1 \leq k \leq n+1$ then $\|g\|_1 \leq \epsilon\nu/(3n)$. We then directly apply Lemma 12 with $\bar{f}(\mathbf{A}) = \frac{\nu}{n}\|\mathbf{A} - \tilde{\mathbf{A}}\|_1$ ($F_{\mathbf{p}}(\mathbf{A}) \geq \frac{\nu}{n}\|\mathbf{A} - \tilde{\mathbf{A}}\|_1$ by Lemma 9) and $\alpha = \epsilon\nu/(3n)$ obtain the desired result. \square